Inverse scattering transform for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions

Xiang-Jun Chen* and Wa Kun Lam†
Department of Physics, Jinan University, Guangzhou 510632, People's Republic of China
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An inverse scattering transform for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions is derived by introducing an affine parameter to avoid constructing Riemann sheets. A one-soliton solution simpler than that in the literature is obtained, which is a breather and degenerates to a bright or dark soliton as the discrete eigenvalue becomes purely imaginary. The solution is mapped to that of the modified nonlinear Schrödinger equation by a gaugelike transformation, predicting some sub-picosecond solitons in optical fibers.

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I. INTRODUCTION

The derivative nonlinear Schrödinger (DNLS) equation

$$iu_t + u_{xx} - im(|u|^2u)_x = 0, \tag{1}$$

where the subscripts denote partial derivatives and $m=\pm 1$, has many physical applications, especially in space plasma physics. It well describes small-amplitude nonlinear Alfvén waves in a low-β (the ratio of kinetic to magnetic pressure) plasma, propagating strictly parallel [1–3] or at a small angle [4,5] to the ambient magnetic field. Recently it was shown that the derivative nonlinear Schrödinger (DNLS) equation also describes large-amplitude magnetohydrodynamic (MHD) waves in a high-β plasma propagating in an arbitrary angle to the ambient magnetic field [6]. Since u represents the complex transverse magnetic field, generally these problems should be modeled with the nonvanishing boundary conditions (NVBC, $|u|\to\text{const}$ as $|x|\to\infty$). The vanishing boundary conditions (VBC, $u\to 0$ as $|x|\to\infty$) can only deal with waves exactly parallel to the ambient field [4,5]. In nonlinear optics, it is well known that the nonlinear Schrödinger (NLS) equation well describes transmission of picosecond pulses in optical fibers [7]. For femtosecond pulse, it was suggested that the nonlinear dispersion term should be included in the NLS equation, resulting in the modified nonlinear Schrödinger (MNLS) equation [8–11] which is related to the DNLS equation by a gaugelike transformation [12]. For problems in optical fibers both VBC and NVBC are of interest [7].

Like other integrable nonlinear equations, soliton dynamics of the DNLS equation is of interest both in theoretical and applied aspects. For the DNLS equation with VBC, one-soliton solution has been found by inverse scattering transform (IST) [13], N-soliton formulas have also been obtained by various approaches [14–16].

Solutions for NVBC problems are much more complicated than those for VBC problems. First, a parameter which is a double-valued function of the eigenvalue usually appears. The IST usually has to be developed on the Riemann sheets of the eigenvalue. Second, there possibly exists a phase shift across the soliton which is relevant to soliton parameters, complicating the derivation of the phase factor of the soliton. In 1978, Kawata and Inoue developed an IST for the DNLS equation with NVBC [17] where they considered the double-valued problem with Riemann sheets. But they only obtained a complicated formula for modulus of the one-soliton solution. With this formula they showed that the solution, generally characterized by two parameters, is a breather-type soliton called paired soliton. When the discrete eigenvalue becomes real (for $m=1$) or purely imaginary (for $m=-1$), the breather reduces to a one-parameter bright or dark soliton, depending on its initial condition [17,18]. One decade later, Mjølhus [4] solved the phase factor of the one-soliton solution (for the case of $m=1$) from formulas provided in Ref. [17] and obtained a rather complicated explicit formula of the two-parameter soliton solution. This solution showed the phase shift across the soliton is zero (or an integer times 2π). When the discrete eigenvalue becomes real it reduces to a simple bright or dark soliton. Behavior of the two-parameter soliton found in Ref. [4] was numerically demonstrated in Ref. [5]. Recently, by using Bäcklund transformation, Steudel [16] derived a formula for N-soliton solution with VBC and NVBC but the explicit expression for the two-parameter one-soliton solution with NVBC is still not simpler than that in Ref. [4].

The physical situations giving rise to exactly solvable equations are highly idealized. Small perturbations violating their integrability, such as Landau damping [5] and density fluctuation [19] for the DNLS equation, actually exist. Perturbation theories for solitons were developed to study effects of small perturbations on soliton transmissions (see, e.g., Refs. [20–22]). For the DNLS solitons with VBC, a direct perturbation theory was recently developed [23], in which the eigenfunctions of the linearized equation around soliton solution were constructed with the squared Jost solutions obtained from the IST [13]. The linearization operator and the way to construct its eigenfunctions with the squared Jost solutions are the same for both VBC and NVBC. Therefore, with results of IST for the DNLS equation with NVBC,

*Electronic address: xiangjun-chen@21cn.com
†Electronic address: wakunlam@21cn.com

in principle, the direct perturbation theory for the DNLS soliton with VBC [23] can be extended to that with NVBC. However, the IST and the soliton solution for the DNLS equation with NVBC in present literature [4,17] seem too complicated to be applied in developing a perturbation theory.

The IST for the DNLS equation with NVBC do have a space for further simplification. It has been shown that the IST with NVBC can become single valued on the plane of an appropriately chosen affine parameter [24]. Some complicated IST problems were greatly simplified by this technique, yielding closed forms of soliton solutions (see, e.g., Refs. [25,26]). In this paper, we develop an IST for the DNLS equation with NVBC by a similar technique. Because the case of $m=1$ can be obtained from the case of $m=-1$ by a transformation $x\to -x$, we just consider the case of $m=-1$. On the plane of the affine parameter, the IST is much simpler than that in Ref. [17]. A much simpler two-parameter one-soliton solution than that in Refs. [4,5] is obtained, which can be easily verified numerically because of its simple form. Although we could not analytically show our solution is identical to that in Refs. [4,5], their numerical behaviors can be shown to be in agreement. When the boundary conditions vanish, both of our solution and that in Ref. [4] approach the solution with VBC [13]. As the discrete eigenvalue becomes purely imaginary, our solution degenerates to a one-parameter bright or dark soliton which is analytically equivalent to that in Ref. [4]. Again, the phase shift across the soliton is shown to be zero, as a direct result of the IST. In the last section, we map the obtained soliton solution to that of the MNLS equation by a gaugelike transform.

II. JOST SOLUTIONS

The Lax equations of Eq. (1) (for $m=-1$) are

$$\partial_x F = LF, \tag{2a}$$

$$\partial_t F = MF, \tag{2b}$$

where the Lax pairs are

$$L = -i\lambda^2\sigma_3 + \lambda U, \tag{3a}$$

$$M = -i2\lambda^4\sigma_3 + 2\lambda^3 U - i\lambda^2 U^2\sigma_3 + \lambda U^3 - i\lambda U_x\sigma_3. \tag{3b}$$

Here

$$U = \begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix}, \tag{4}$$

$\sigma_i (i=1,2,3)$ are Pauli matrices, the bar stands for complex conjugate, and λ is the time-independent eigenvalue. Without loss of generality, the NVBC can be written as

$$u \to \rho e^{\pm i2\alpha} \quad \text{as} \quad x \to \pm\infty. \tag{5}$$

Here ρ is real and we have assumed that there is a phase shift of 4α across the soliton. In asymptotic solutions of Eq. (2a) as $|x|\to\infty$, a double-valued function of λ, $\zeta=(\lambda^2+\rho^2)^{1/2}$ appears. Introducing an affine parameter [24] k satisfying

$$\lambda = \tfrac{1}{2}(k - \rho^2 k^{-1}), \tag{6}$$

we have

$$\zeta = \tfrac{1}{2}(k + \rho^2 k^{-1}), \tag{7}$$

which is a single-valued function of k. Asymptotic solutions of Eq. (2a) are

$$E^{\pm}(x,k) = e^{\pm i\alpha\sigma_3}(I - i\rho k^{-1}\sigma_1)e^{-i\lambda\zeta x\sigma_3} \quad \text{as } x \to \pm\infty, \tag{8}$$

where I is the unit matrix. As usual, we define Jost solutions which have the following asymptotic behaviors

$$\Psi(x,k) \to E^{+}(x,k) \quad \text{as} \quad x \to \infty, \tag{9a}$$

$$\Phi(x,k) \to E^{-}(x,k) \quad \text{as} \quad x \to -\infty, \tag{9b}$$

and the scattering coefficients by

$$\Phi(x,k) = \Psi(x,k)T(k), \tag{10}$$

where

$$\Psi(x,k) = [\tilde{\psi}(x,k), \psi(x,k)], \tag{11a}$$

$$\Phi(x,k) = [\phi(x,k), \tilde{\phi}(x,k)], \tag{11b}$$

$$T(k) = \begin{pmatrix} a(k) & -\tilde{b}(k) \\ b(k) & \tilde{a}(k) \end{pmatrix}. \tag{12}$$

Equation (10) yields

$$a(k) = \det(\phi,\psi)/\det\Psi, \tag{13a}$$

$$\tilde{a}(k) = \det(\tilde{\psi},\tilde{\phi})/\det\Psi, \tag{13b}$$

and

$$\det\Phi = \det\Psi\,\det T. \tag{14}$$

From Eq. (2), we have $\partial_x \det\Psi(x,k)=0$ and $\partial_x \det\Phi(x,k)=0$, Therefore

$$\det\Psi(x,k) = \det E^{+}(x,k) = 1 + \rho^2 k^{-2}, \tag{15a}$$

$$\det\Phi(x,k) = \det E^{-}(x,k) = 1 + \rho^2 k^{-2}, \tag{15b}$$

$$\det T(k) = 1. \tag{16}$$

III. SYMMETRIES ON THE k PLANE

The key to simplify the IST is to find symmetries of the Jost solutions and the scattering coefficients on the k plane. If one finds that $L(x,k)$ is unchanged upon a transformation, for example,

$$\sigma_2\overline{L(x,\overline{k})}\sigma_2 = L(x,k), \tag{17}$$

then, if $F(x,k)$ is a solution of Eq. (2a), we have

$$\partial_x \sigma_2 \overline{F(x,\overline{k})} \sigma_2 = L(x,k) \sigma_2 \overline{F(x,\overline{k})} \sigma_2. \tag{18}$$

Thus, $\sigma_2 \overline{F(x,\overline{k})} \sigma_2$ or $\sigma_2 \overline{F(x,\overline{k})} \sigma_2 \sigma_1$ is also a solution of Eq. (2a), corresponding to the same eigenvalue k. They only differ in a constant factor. If $F(x,k)$ is a Jost solution, this constant factor can be determined by its definite asymptotic behavior. We find

$$\sigma_2 \overline{E^{\pm}(x,\overline{k})} \sigma_2 = E^{\pm}(x,k), \tag{19}$$

hence

$$\sigma_2 \overline{\Psi(x,\overline{k})} \sigma_2 = \Psi(x,k), \quad \sigma_2 \overline{\Phi(x,\overline{k})} \sigma_2 = \Phi(x,k), \tag{20}$$

and, with Eq. (10), we get

$$\sigma_2 \overline{T(\overline{k})} \sigma_2 = T(k), \tag{21}$$

that is,

$$\tilde{\psi}(x,k) = i\sigma_2 \overline{\psi(x,\overline{k})}, \quad \tilde{\phi}(x,k) = -i\sigma_2 \overline{\phi(x,\overline{k})}, \tag{22}$$

$$\tilde{a}(k) = \overline{a(\overline{k})}. \tag{23}$$

The second symmetric relation is upon $k \to -k$. We find

$$\sigma_3 L(x,-k) \sigma_3 = L(x,k), \tag{24}$$

$$\sigma_3 E^{\pm}(x,-k) \sigma_3 = E^{\pm}(x,k). \tag{25}$$

Therefore

$$\sigma_3 \Psi(x,-k) \sigma_3 = \Psi(x,k), \quad \sigma_3 \Phi(x,-k) \sigma_3 = \Phi(x,k), \tag{26}$$

$$\sigma_3 T(-k) \sigma_3 = T(k). \tag{27}$$

That is,

$$\psi(-k) = -\sigma_3 \psi(k), \quad \tilde{\psi}(-k) = \sigma_3 \tilde{\psi}(k), \tag{28a}$$

$$\phi(-k) = \sigma_3 \phi(k), \quad \tilde{\phi}(-k) = -\sigma_3 \tilde{\phi}(k), \tag{28b}$$

$$a(-k) = a(k). \tag{29}$$

The third symmetric relation is upon $k \to \rho^2 k^{-1}$. We find

$$\sigma_3 L(x,\rho^2 k^{-1}) \sigma_3 = L(x,k), \tag{30}$$

$$\sigma_3 E^{\pm}(x,\rho^2 k^{-1}) \sigma_3 \sigma_1 = i\rho^{-1} k E^{\pm}(x,k). \tag{31}$$

Therefore

$$\Psi(x,\rho^2 k^{-1}) = \rho^{-1} k \sigma_3 \Psi(x,k) \sigma_2, \quad \Phi(x,\rho^2 k^{-1}) = \rho^{-1} k \sigma_3 \Phi(x,k) \sigma_2, \tag{32}$$

$$T(\rho^2 k^{-1}) = \sigma_2 T(k) \sigma_2 = \overline{T(\overline{k})}. \tag{33}$$

That is,

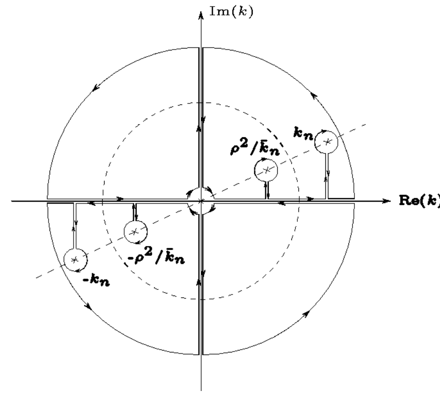

FIG. 1. Integration path for Eq. (55). The radius of the dashed circle is ρ. The radius of the large solid circle approaches ∞.

$$\tilde{\psi}(x,\rho^2 k^{-1}) = i\rho^{-1} k \sigma_3 \psi(x,k), \quad \tilde{\phi}(x,\rho^2 k^{-1}) = -i\rho^{-1} k \sigma_3 \phi(x,k), \tag{34}$$

$$a(\rho^2 k^{-1}) = \tilde{a}(k) = \overline{a(\overline{k})}. \tag{35}$$

Thus, as shown in Fig. 1, if $k_{n1}=k_n$ is a simple zero of $a(k)$ in the first quadrant, outside the ρ circle, $k_{n2}=-k_n$ is also a simple zero in the third quadrant outside the ρ circle, $k_{n3}=\rho^2 \overline{k}_n^{-1}$ is also a simple zero in the first quadrant inside the ρ circle, and $k_{n4}=-\rho^2 \overline{k}_n^{-1}$ is also a simple zero in the fourth quadrant inside the ρ circle. Correspondingly, $\overline{k_{nj}}$ ($j=1,2,3,4$) are zeros of $\tilde{a}(k)$. Equation (13a) yields

$$\phi(x,k_{nj}) = b_{nj} \psi(x,k_{nj}), \quad \tilde{\phi}(x,\overline{k_{nj}}) = \tilde{b}_{nj} \tilde{\psi}(x,\overline{k_{nj}}). \tag{36}$$

Equation (20) yields

$$\tilde{b}_{nj} = -\overline{b_{nj}}. \tag{37}$$

Because of Eqs. (28) and (34), we get

$$b_{n2} = -b_n, \quad b_{n3} = \overline{b_n}, \quad b_{n4} = -\overline{b_n}. \tag{38}$$

Equations (29) and (35) yield

$$\dot{a}(k_{n2}) = -\dot{a}(k_n), \quad \dot{a}(k_{n3}) = -\rho^{-2} \overline{k_n^2 \dot{a}(k_n)}, \quad \dot{a}(k_{n4}) = \rho^{-2} \overline{k_n^2 \dot{a}(k_n)}. \tag{39}$$

Letting

$$c_{nj} = \frac{b_{nj}}{\dot{a}(k_{nj})}, \tag{40}$$

we have

$$c_{n2} = c_{n1} = c_n, \tag{41a}$$

$$c_{n3} = c_{n4} = -\rho^2 \overline{k}_n^{-2} \overline{c_n}. \tag{41b}$$

IV. ASYMPTOTIC BEHAVIORS ON THE k PLANE

In the first and the third quadrants of k plane, $\mathrm{Im}(\lambda\zeta) \geqslant 0$, $\psi(x,k)$, $\phi(x,k)$, and $a(k)$ are analytic. In the second and the fourth quadrants of k plane, $\mathrm{Im}(\lambda\zeta) \leqslant 0$, $\tilde{\psi}(x,k)$, $\tilde{\phi}(x,k)$, and $\tilde{a}(k)$ are analytic [17]. As usual, asymptotic behaviors of Jost solutions as $|k| \to \infty$ or $k \to 0$ can be found from Eq. (2a). Asymptotic behaviors of $a(k)$ and $\tilde{a}(k)$ are then obtained from Eq. (13a).

As $|k| \to \infty$, we have

$$\psi(x,k)e^{-i\lambda\zeta x} \to \begin{pmatrix} -ik^{-1}u \\ 1 \end{pmatrix} e^{i(\eta^+ - \alpha)} + O(|k|^{-2}), \tag{42a}$$

$$\phi(x,k)e^{i\lambda\zeta x} \to \begin{pmatrix} 1 \\ -ik^{-1}\overline{u} \end{pmatrix} e^{i(\eta^- - \alpha)} + O(|k|^{-2}), \tag{42b}$$

and, by using Eq. (13a),

$$a(k) \to \exp(i\eta - i2\alpha). \tag{43}$$

Here

$$\eta^{\pm} = \pm \frac{1}{2}\int_x^{\pm\infty} (\rho^2 - |u|^2)dx, \tag{44}$$

$$\eta = \eta^+ + \eta^- = \frac{1}{2}\int_{-\infty}^{+\infty} (\rho^2 - |u|^2)dx. \tag{45}$$

As $k \to 0$, we have

$$\psi(x,k)e^{-i\lambda\zeta x} \to \begin{pmatrix} -i\rho k^{-1} \\ \rho^{-1}\overline{u} \end{pmatrix} e^{-i(\eta^+ - \alpha)} + O(1), \tag{46a}$$

$$\phi(x,k)e^{i\lambda\zeta x} \to \begin{pmatrix} \rho^{-1}u \\ -i\rho k^{-1} \end{pmatrix} e^{-i(\eta^- - \alpha)} + O(1), \tag{46b}$$

and, by using Eq. (13a), we have

$$a(k) \to \exp[-i(\eta - 2\alpha)]. \tag{47}$$

With analytic and asymptotic behaviors of $a(k)$ and $\tilde{a}(k)$, usual IST procedure yields

$$a(k) = \exp[i(\eta - 2\alpha)]\prod_{n=1}^{N}\prod_{j=1}^{4}\frac{k-k_{nj}}{k-\overline{k_{nj}}}\exp\left[\frac{1}{2\pi i}\int_{\Gamma}\frac{\ln|a(k')|^2}{k'-k}dk'\right]. \tag{48}$$

Here Γ is the path consisting of lines from $i\infty$ to 0, from 0 to ∞, from $-i\infty$ to 0, and from 0 to $-\infty$.

As $k \to 0$, Eq. (48) becomes

$$a(k) \to \exp\left[i(\eta - 2\alpha) + i8\sum_n \beta_n\right], \quad \beta_n = \arg(k_n). \tag{49}$$

Comparing Eq. (49) with Eq. (47), we get a relation between the phase shift and the soliton parameters:

$$4\alpha = 2\eta + 8\sum_{n=1}^{N} \beta_n. \tag{50}$$

V. ZAKHAROV-SHABAT INVERSE SCATTERING EQUATION

For the case when $a(k)$ only has simple zeros, we define

$$\Theta(x,k) = \begin{cases} a^{-1}(k)\phi(x,k) & \text{in the first and third quadrants of } k \\ \tilde{\psi}(x,k) & \text{in the second and fourth quadrants of } k. \end{cases} \tag{51}$$

With Eqs. (42), (46), and (22), we get

$$\Theta(x,k)e^{i\lambda\zeta x} \to e^{-i(\eta^+ - \alpha)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{as } |k| \to \infty \tag{52}$$

and

$$\Theta(x,k)e^{i\lambda\zeta x} \to e^{i(\eta^+ - \alpha)}\begin{pmatrix} \rho^{-1}u \\ -i\rho k^{-1} \end{pmatrix} \quad \text{as } |k| \to 0. \tag{53}$$

Thus, besides the $4N$ poles correspond to zeros of $a(k)$, $\Theta(x,k)$ has an extra pole at $k=0$. While across the real or imaginary axes, $\Theta(x,k)$ has a jump,

$$a^{-1}(k)\phi(x,k) - \tilde{\psi}(x,k) = r(k)\psi(x,k), \tag{54}$$

where $r(k)=b(k)/a(k)$ is the reflection coefficient. For k interior to the closed path shown in Fig. 1, Cauchy's formula yields

$$\Theta(x,k)e^{i\lambda\zeta x} - e^{-i(\eta^+ - \alpha)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{i2\pi}\oint \frac{\Theta(x,k')e^{i\lambda'\zeta' x} - e^{-i(\eta^+ - \alpha)}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{k'-k}dk' = R(x,k) + J(x,k), \tag{55}$$

where

$$\begin{aligned} R(x,k) &= -\sum_{n=1}^{N}\sum_{j=1}^{4}\mathrm{Res}\left[\frac{\Theta(x,k')e^{i\lambda'\zeta' x}}{(k'-k)}, k'=k_{nj}\right] - \mathrm{Res}\left[\frac{\Theta(x,k')e^{i\lambda'\zeta' x}}{(k'-k)}, k'=0\right] \\ &= \sum_{n=1}^{N}\sum_{j=1}^{4}\frac{1}{(k-k_{nj})}c_{nj}\psi(x,k_{nj})e^{i\lambda_{nj}\zeta_{nj}x} - i\rho k^{-1}e^{i(\eta^+ - \alpha)}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{56}$$

and

$$J(x,k) = \frac{1}{i2\pi}\int_{\Gamma}\frac{r(k')\psi(x,k')e^{i\lambda'\zeta' x}}{k'-k}dk'. \tag{57}$$

Hence

$$\Theta(x,k)e^{i\lambda\zeta x} = \begin{pmatrix} e^{-i(\eta^+-\alpha)} \\ -i\rho k^{-1}e^{i(\eta^+-\alpha)} \end{pmatrix} + \sum_{n=1}^{N}\sum_{j=1}^{4} \frac{1}{k-k_{nj}} c_{nj}\psi(x,k_{nj})e^{i\lambda_{nj}\zeta_{nj}x} + J(x,k). \tag{58}$$

For k in second or fourth quadrants, we have the Zakharov-Shabat inverse scattering equation [27]

$$\tilde{\psi}(x,k)e^{i\lambda\zeta x} = \begin{pmatrix} e^{-i(\eta^+-\alpha)} \\ -i\rho k^{-1}e^{i(\eta^+-\alpha)} \end{pmatrix} + \sum_{n=1}^{N}\sum_{j=1}^{4} \frac{1}{k-k_{nj}} c_{nj}\psi(x,k_{nj})e^{i\lambda_{nj}\zeta_{nj}x} + J(x,k). \tag{59}$$

For the case of reflectionless potential, $J(x,k)=0$, we can find Jost solutions from Eq. (59). Soliton solutions can be found from Eq. (42) or Eq. (46), e.g., Eqs. (46a) and (22) yield

$$u(x) = \rho e^{-i(\eta^+-\alpha)} \lim \tilde{\psi}_1(x,k)e^{i\lambda\zeta x} = \rho e^{-i2(\eta^+-\alpha)} - \rho e^{-i(\eta^+-\alpha)} \sum_{n=1}^{N}\sum_{j=1}^{4} \frac{c_{nj}}{k_{nj}}\psi_1(x,k_{nj})e^{i\lambda_{nj}\zeta_{nj}x}.$$

Considering Eqs. (28), Eq. (34) and (41), we get

$$u(x) = \rho e^{-i2(\eta^+-\alpha)} - 2\rho e^{-i(\eta^+-\alpha)} \sum_{n=1}^{N} \left[\frac{c_n}{k_n}\psi_1(x,k_n)e^{i\lambda_n\zeta_n x} + i\frac{\bar{c}_n}{\rho}\overline{\psi_2(x,k_n)}e^{-i\bar{\lambda}_n\bar{\zeta}_n x} \right]. \tag{60}$$

In order that the Jost solutions obtained from the first Lax equation satisfy the second Lax equation, they must be multiplied by a t-dependent factor as

$$\tilde{\psi}(x,k,t) \to h(t,k)\tilde{\psi}(x,k), \quad \psi(x,k,t) \to h^{-1}(t,k)\psi(x,k),$$

$$\phi(x,k,t) \to h(t,k)\phi(x,k), \quad \tilde{\phi}(x,k,t) \to h^{-1}(t,k)\tilde{\phi}(x,k).$$

With standard IST techniques, we find

$$h(t,k) = \exp[-i\lambda\zeta(2\lambda^2-\rho^2)t], \tag{61}$$

and the t dependence of all scattering data,

$$a(k,t) = a(k,0), \tag{62}$$

$$b(k,t) = b(k,0)\exp[i2\lambda\zeta(2\lambda^2-\rho^2)t], \tag{63}$$

$$b_{nj}(t) = b_{nj}(0)\exp[i2\lambda_{nj}\zeta_{nj}(2\lambda_{nj}^2-\rho^2)t]. \tag{64}$$

Therefore

$$c_{nj}(t) = c_{nj}(0)\exp[i2\lambda_{nj}\zeta_{nj}(2\lambda_{nj}^2-\rho^2)t]. \tag{65}$$

VI. ONE-SOLITON SOLUTION

For the case of $N=1$, with Eqs. (28), (34), and (41), Eq. (59) becomes

$$\tilde{\psi}(x,k)e^{i\lambda\zeta x} = \begin{pmatrix} e^{-i(\eta^+-\alpha)} \\ -i\rho k^{-1}e^{i(\eta^+-\alpha)} \end{pmatrix} + \frac{2}{k^2-k_1^2}\begin{pmatrix} k_1 & 0 \\ 0 & k \end{pmatrix} c_1\psi(x,k_1)e^{i\lambda_1\zeta_1 x} + \frac{i2\rho\bar{k}_1^{-1}}{(k^2-\rho^4\bar{k}_1^{-2})}\begin{pmatrix} 0 & \rho^2\bar{k}_1^{-1} \\ k & 0 \end{pmatrix} \bar{c}_1\overline{\psi(x,k_1)}e^{-i\bar{\lambda}_1\bar{\zeta}_1 x}. \tag{66}$$

At $k=\overline{k_{11}}=\overline{k_1}$, with Eq. (22), we have

$$\psi_1(x,k_1)e^{-i\lambda_1\zeta_1 x} = -i\rho k_1^{-1}e^{-i(\eta^+-\alpha)} - \frac{2k_1}{k_1^2-\bar{k}_2^1}\bar{c}_1\overline{\psi_2(x,k_1)}e^{-i\bar{\lambda}_1\bar{\zeta}_1 x} + \frac{i\rho}{2\lambda_1\zeta_1}c_1\psi_1(x,k_1)e^{i\lambda_1\zeta_1 x}, \tag{67a}$$

$$\overline{\psi_2(x,k_1)}e^{i\bar{\lambda}_1\bar{\zeta}_1 x} = e^{-i(\eta^+-\alpha)} - \frac{2k_1}{k_1^2-\bar{k}_2^1}c_1\psi_1(x,k_1)e^{i\lambda_1\zeta_1 x} + \frac{i\rho^3\bar{k}_1^{-2}}{2\bar{\lambda}_1\bar{\zeta}_1}\bar{c}_1\overline{\psi_2(x,k_1)}e^{-i\bar{\lambda}_1\bar{\zeta}_1 x}. \tag{67b}$$

It is easy to verify that for $j=2,3,4$, we just have the same equations or their complex conjugate. Letting

$$k_1 = \rho e^{\gamma_1+i\beta_1}, \quad \gamma_1 \geqslant 0, \quad 0 < \beta_1 < \pi/2, \tag{68}$$

$$2\lambda_1\zeta_1 = \mu + i\nu, \tag{69}$$

$$\mu = \rho^2\sinh(2\gamma_1)\cos(2\beta_1), \quad \nu = \rho^2\cosh(2\gamma_1)\sin(2\beta_1), \tag{70}$$

$$c_1(0) = -i\frac{2\lambda_1\zeta_1}{\bar{k}_1}\sin(2\beta_1)e^{\nu x_0+i\varphi_0}, \tag{71}$$

solving Eq. (67) and substituting its solutions into Eq. (60), we get

$$u(x,t) = \rho e^{-i(2\eta^+-\eta)}\frac{N}{D}, \tag{72}$$

where

$$N = e^{\theta+i3\beta_1} + \sinh^2(2\gamma_1)e^{-4\gamma_1}e^{-\theta-i3\beta_1} + \sin(2\beta_1)e^{\gamma_1-i\varphi} - \sin(2\beta_1)e^{-5\gamma_1+i\varphi}, \tag{73}$$

$$D = e^{\theta-i\beta_1} + \sinh^2(2\gamma_1)e^{-4\gamma_1}e^{-\theta+i\beta_1} + \sin(2\beta_1)e^{-3\gamma_1-i\varphi} - \sin(2\beta_1)e^{-\gamma_1+i\varphi}, \tag{74}$$

$$\theta = \nu(x-vt-x_0), \quad \varphi = \mu(x-wt)+\varphi_0, \tag{75}$$

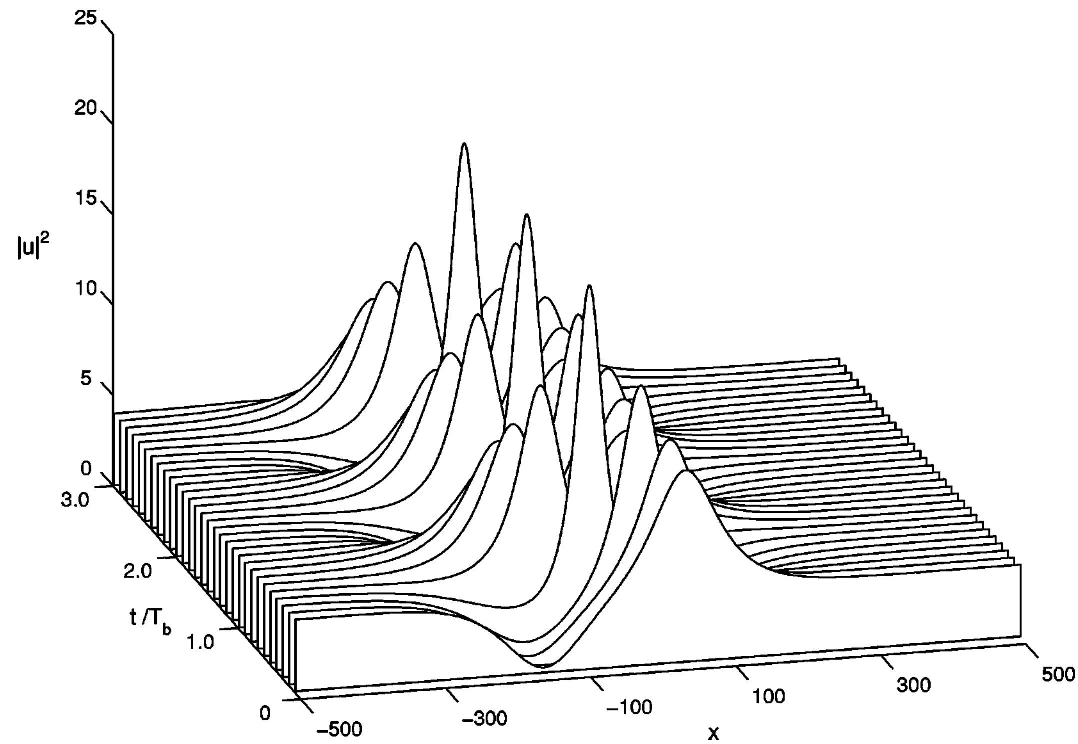

FIG. 2. Time evolution of the breather, Eq. (82), in three periods, where $\rho=2$, $\gamma_1=0.08$, $\beta_1=\pi/10$, $x_0=0$, and $\varphi_0=3\pi/2$.

$$v = 2\rho^2 - \rho^2\cos(2\beta_1)\frac{\cosh(4\gamma_1)}{\cosh(2\gamma_1)}, \tag{76}$$

$$w = 2\rho^2 - \rho^2\cosh(2\gamma_1)\frac{\cos(4\beta_1)}{\cos(2\beta_1)}. \tag{77}$$

In order to determine $\eta^+(x)$ with Eq. (44), we find a useful relation

$$D_1\frac{dD_2}{dx} - D_2\frac{dD_1}{dx} = \frac{\rho^2}{4}(|D|^2 - |N|^2), \tag{78}$$

where $D_1=\mathrm{Re}\,D$, $D_2=\mathrm{Im}\,D$. With this relation, we have

$$\eta^+(x) = 2\int_x^\infty \frac{1}{1+(D_2/D_1)^2}\frac{d}{dx}(D_2/D_1) = -i\ \ln(D/\overline{D})|_x^\infty$$
$$= i\ln\frac{D}{\overline{D}} - 2\beta_1 \tag{79}$$

and

$$\eta = \eta^+(-\infty) = -4\beta_1, \tag{80}$$

$$\alpha = \frac{\eta}{2} + 2\beta_1 = 0. \tag{81}$$

This means there is no phase shift across the soliton. Therefore we have the one-soliton solution

$$u = \rho\frac{N\overline{D}}{D^2}. \tag{82}$$

In general case λ_1 is complex. There are two soliton parameters γ_1 and β_1 characterizing its behavior, and the solution is usually called two-parameter soliton in the literature [5]. It is actually a breather with a period,

$$T_b = \frac{2\pi}{\mu|v-w|} = \frac{2\pi}{\rho^2\tanh(2\gamma_1)[\cosh^2(2\gamma_1)+\cos^2(2\beta_1)]}. \tag{83}$$

We have numerically verified that the breather, Eq. (82), really satisfies Eq. (1) ($m=-1$). Its time evolution in three periods is shown in Fig. 2, in agreement with that numerically exhibited in Ref. [5].

Equation (82) includes the VBC as a special case. As $\rho \to 0$, $\gamma_1 \to +\infty$, keeping $\rho e^{\gamma_1} = |k_1| = 2|\lambda_1|$ finite, we have

$$\rho N \to |k_1|\sin(2\beta_1)e^{-i\varphi}, \tag{84}$$

$$D \to e^{\theta - i\beta_1} + \tfrac{1}{4}e^{-\theta + i\beta_1}. \tag{85}$$

Redefining x_0 by absorbing $-\ln 2/\nu$ into x_0, we get the one-soliton solution with VBC,

$$u = \frac{4|\lambda_1|\sin(2\beta_1)e^{-i\varphi}}{(e^{\theta+i\beta_1}+e^{-\theta-i\beta_1})^2}(e^{\theta-i\beta_1}+e^{-\theta+i\beta_1}), \tag{86}$$

which is equivalent to that in Ref. [13].

For a special case when k_1 approaches the ρ circle, i.e., λ_1 becomes purely imaginary, $\gamma_1 \to 0$, $\mu \to 0$, $\varphi \to \varphi_0$. There is only one soliton parameter β_1 characterizing the soliton. It is thus called one-parameter soliton [5]. If $\varphi_0 \neq n\pi$ (n is an integer), we have

$$u(x,t) = \rho\left[1 - \frac{i4\epsilon \cos^2 \beta_1}{e^{\theta+i\beta_1} - e^{-\theta-i\beta_1} + i2\epsilon}\right], \tag{87}$$

where x_0 has been redefined by absorbing $\ln[2 \sin(2\beta_1)|\sin \varphi_0|]/\nu$ into x_0, $\epsilon = \mathrm{sgn}(\sin \varphi_0)$. The case $\epsilon = -1(1)$ corresponds to bright(dark) soliton, equivalent to those obtained in Ref. [4].

VII. ONE-SOLITON SOLUTION FOR THE MNLS EQUATION WITH NVBC

The MNLS equation [8] is

$$iq_z + \frac{\sigma}{2} q_{tt} + is(|q|^2 q)_t + |q|^2 q = 0, \tag{88}$$

where $\sigma = \pm 1$, $\sigma = -1(1)$ corresponds to normal (abnormal) group velocity dispersion (GVD) region, the third term on the left is the nonlinear dispersion term, while s represents its relative magnitude [7]. It can be verified that upon a gauge-like transformation

$$q(z,t) = Q(Z,T) e^{i(1/4)s^{-4}Z + i(\sigma/2)s^{-2}T}, \tag{89}$$

in which

$$t = \frac{\sigma}{2} s^{-1} T + \frac{1}{2} s^{-3} Z, \quad z = \frac{\sigma}{2} s^{-2} Z, \tag{90}$$

the MNLS equation, Eq. (88), is transformed to the DNLS equation

$$iQ_Z + Q_{TT} + i(|Q|^2 Q)_T = 0. \tag{91}$$

From one-soliton solution of the DNLS equation obtained in the preceding section, we have one-soliton solution of the MNLS equation with NVBC,

$$q(z,t) = \rho \frac{N\bar{D}}{D^2} e^{-i(\sigma/2)s^{-2}z + is^{-1}t}. \tag{92}$$

Here dependence of N and D on θ and φ are the same as Eqs. (73) and (74), while

$$\theta = 2\sigma s \nu [t - (sv + \sigma s^{-1})z - t_0], \tag{93}$$

$$\varphi = 2\sigma s \mu [t - (sw + \sigma s^{-1})z] + \varphi_0. \tag{94}$$

As $\gamma_1 \to 0$, we also get

$$q(z,t) = \rho\left[1 - \frac{i4\epsilon \cos^2 \beta_1}{e^{\theta+i\beta_1} - e^{-\theta-i\beta_1} + i2\epsilon}\right] e^{-i(\sigma/2)s^{-2}z + is^{-1}t}, \tag{95}$$

Therefore, when effects of the nonlinear dispersion are significant enough, with background wave, single-mode fibers may possibly support breathers, bright solitons, and dark solitons, both in the regions of normal and abnormal GVD. Here, the MNLS dark soliton is very different from its NLS counterpart. The MNLS dark soliton not only exists in normal but also in abnormal GVD region while the NLS dark soliton only exists in normal GVD region. There is no phase shift across the MNLS dark soliton but there is a phase shift across the NLS dark soliton which is relevant to soliton parameters (see, e.g., Ref. [28]).

VIII. SUMMARY AND DISCUSSION

In this paper, by introducing an affine parameter, a simple IST for the DNLS equation with NVBC is developed, yielding a much simpler one-soliton solution than that in the literature. We show that the DNLS equation with NVBC supports rich soliton dynamics. It supports breathers which look like bounded pairs of bright and dark solitons, as well as unpaired bright and dark solitons. Our solution includes the case of VBC as a special case. As the solution is mapped to that for the MNLS equation with NVBC, it predicts some solitons in optical fibers in sub-picosecond regime with background waves, especially, a completely different type of dark solitons which has no phase shift across itself and may exist not only in normal GVD region but also in abnormal GVD region. The IST technique developed in this paper make it possible to get multisoliton solutions for the DNLS/MNLS equation. When multisoliton solutions are obtained, they will demonstrate interesting soliton dynamics such as collisions between bright and dark solitons and collisions between breather and bright or dark solitons. The IST technique also provides a foundation for understanding effects of small perturbations on the DNLS/MNLS solitons with NVBC. There seems no special difficulty to extend the direct perturbation theory for the DNLS/MNLS solitons with VBC [23] to that with NVBC.

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